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Four dimensional $\mathcal{R}^4$ superinvariants through gauge completion

Filipe Moura

C.N. Yang Institute for Theoretical Physics, State University of New York
Stony Brook, NY 11794-3840, USA
E-mail: fmoura@insti.physics.sunysb.edu

Abstract: We fully compute the $\mathcal{N}=1$ supersymmetrization of the fourth power of the Weyl tensor in $d=4$ x-space with the auxiliary fields. In a previous paper, we showed that their elimination requires an infinite number of terms; we explicitely compute those terms to order κ^4 (three loop). We also write, in superspace notation, all the possible $\mathcal{N}=1$ actions, in four dimensions, that contain pure $\mathcal{R}^4$ terms (with coupling constants). We explicitely write these actions in terms of the θ components of the chiral density ϵ and the supergravity superfields R, G_m, W_{ABC}. Using the method of gauge completion, we compute the necessary θ components which allow us to write these actions in x-space. We discuss under which circumstances can these extra $\mathcal{R}^4$ correction terms be reabsorbed in the pure supergravity action, and their relevance to the quantum supergravity/string theory effective actions.

Keywords: Supergravity Models, Supersymmetric Effective Theories, Superspaces.

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Contents

1. Introduction

In a previous paper [1], we wrote and analyzed in $d=4, \mathcal{N}=1$ superspace a lagrangian which contained a pure $\mathcal{R}^4$ term:

$$\mathcal{L} = \frac{1}{2\kappa^2}\int E\left(1+\alpha W^2\overline{W}^2\right)d^4\theta \tag{1.1}$$

This lagrangian could be seen as a four dimensional $\mathcal{N}=1$ string/M-theory effective action, resulting from a compactification and truncation from ten/eleven dimensions. In that case the constant α/κ^2 could be identified up to a factor with α'^3, α' being the string slope parameter.

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Originally the $\alpha W^2\overline{W}^2$ term of this lagrangian was thought of as a possible three-loop quantum correction to supergravity. In reference [2] it was shown that supersymmetry implies the one and two-loop finiteness of supergravity, but there exists a potential non-vanishing on-shell superinvariant which could mean non-finiteness of supergravity at three loops. Later that superinvariant was written in superspace [3, 4]: it is precisely the α part of (1.1). No one has been able to compute the three-loop supergravity effective action in order to show if the coefficient of this term is nonzero or if it vanishes by some miraculous cancellation [5]. Up to that (unknown) numerical factor, the constant α could be identified with κ^6 if (1.1) is that effective action.

In the paper [1], we made a full superspace analysis of (1.1). In section 2 of this paper, we extend this analysis to x-space. We write down (1.1) in components, leaving the auxiliary fields. As we have shown in [1], the auxiliary fields do not vanish anymore with this action; the elimination of the A_m auxiliary field would result in a nonlocal action with an infinite number of terms of all orders in α. We fully compute the terms which are of first order in α and which could constitute the three-loop supergravity effective action.

In section 3 we argue that there are other supersymmetric pure $\mathcal{R}^4$ terms which can be considered. These terms can be absorbed in the pure supergravity action by a field redefinition, but this process induces new higher-order superinvariants. We list these other supersymmetric $\mathcal{R}^4$ terms, compute their bosonic parts and comment on their properties.

2. The $\mathcal{W}_+^2\mathcal{W}_-^2$ supersymmetric action in x space

The lagrangian (1.1) represents the supersymmetrization of one combination of $\mathcal{W}^4$, namely $\mathcal{W}_+^2\mathcal{W}_-^2$ (for notation see appendix A). In order to write it in components, first we write it in chiral notation:

$$\mathcal{L} = \frac{1}{4\kappa^2}\int \epsilon\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left(-3\alpha W^2\overline{W}^2 - 3\right)\right] d^2\theta + \text{h.c.} \tag{2.1}$$

ϵ is the chiral density; its expansion in components is [6]

$$2\epsilon = e - ie\theta^A\sigma_{mA\dot{A}}\psi^{m\dot{A}} - e\theta^2\left(M - iN - \frac{1}{2}\psi^{m\dot{A}}\psi^{n\dot{B}}\sigma_{mn\dot{A}\dot{B}}\right) \tag{2.2}$$

We get, then, for the lagrangian,

$$\begin{aligned}
\kappa^2\mathcal{L} = & -\frac{1}{2}e\mathcal{R} + \frac{1}{4}e\varepsilon^{\mu\nu\rho\lambda}\left(\psi_{\mu\dot{A}}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda A} - \psi_{\mu A}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda\dot{A}}\right) - \frac{1}{3}e\left(M^2 + N^2 - A^\mu A_\mu\right) - \\
& - \frac{3}{16}e\alpha\, \nabla^2 W^2\big|\, \overline{\nabla}^2\overline{W}^2\Big| - \frac{\alpha}{2}e\mathcal{R}\, W^2\big|\, \overline{W}^2\Big| + 3\alpha e\partial_\mu\, W^2\big|\, \partial^\mu\, \overline{W}^2\Big| + \\
& + \frac{\alpha}{4}e\, W^2\big|\, \overline{W}^2\Big|\, \varepsilon^{\mu\nu\rho\lambda}\left(\psi_{\mu\dot{A}}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda A} - \psi_{\mu A}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda\dot{A}}\right) - \\
& - \frac{3}{4}i\alpha e\left(\nabla^A W^2\big|\, \sigma^\mu_{A\dot{A}}\mathcal{D}_\mu\, \nabla^{\dot{A}}\overline{W}^2\Big| + \nabla^{\dot{A}}\overline{W}^2\Big|\, \sigma^\mu_{A\dot{A}}\mathcal{D}_\mu\, \nabla^A W^2\big|\right) + \\
& + \frac{3}{4}\alpha e\varepsilon^{\mu\nu\rho\lambda}\psi_{\mu A}\sigma_\nu^{A\dot{A}}\psi_{\rho\dot{A}}\left(\overline{W}^2\Big|\, \partial_\lambda\, W^2\big| - W^2\big|\, \partial_\lambda\, \overline{W}^2\Big|\right) - \\
& - \frac{3}{2}\alpha e\left(\psi^{\mu A}\, \nabla_A W^2\big| + \sigma^{\mu\nu}_{AB}\psi_\nu^A\, \nabla^B W^2\big|\right)\partial_\mu\, \overline{W}^2\Big| -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{2}\alpha e\left(\psi^{\mu\dot{A}}\;\nabla_{\dot{A}}\overline{W}^2\Big| - \sigma^{\mu\nu}_{\dot{A}\dot{B}}\psi^{\dot{A}}_{\nu}\;\nabla^{\dot{B}}\overline{W}^2\Big|\right)\partial_\mu\,W^2\big| + \\
&+\frac{\alpha}{2}e\left(\overline{W}^2\Big|\,\sigma^{\mu\nu}_{AB}\;\nabla^A W^2\big|\,\psi^B_{\mu\nu} + W^2\big|\,\sigma^{\mu\nu}_{\dot{A}\dot{B}}\;\nabla^{\dot{A}}\overline{W}^2\Big|\,\psi^{\dot{B}}_{\mu\nu}\right) - \\
&-\frac{3}{4}\alpha e\;\nabla^A W^2\big|\;\nabla^{\dot{A}}\overline{W}^2\Big|\left(\psi_{\mu A}\psi^{\mu}_{\dot{A}} + \frac{1}{2}\sigma^{\mu\nu}_{\dot{A}\dot{B}}\psi_{\mu A}\psi^{\dot{B}}_{\nu} + \frac{1}{2}\sigma^{\mu\nu}_{AB}\psi^B_{\mu}\psi_{\nu\dot{A}}\right) + \\
&+\frac{3}{8}\alpha e\varepsilon^{\mu\nu\rho\lambda}\sigma_{\lambda A\dot{A}}\left(\overline{W}^2\Big|\,\psi^A_{\mu}\psi^B_{\nu}\psi^{\dot{A}}_{\rho}\;\nabla_B W^2\big| - W^2\big|\,\psi^A_{\mu}\psi^{\dot{A}}_{\nu}\psi^{\dot{B}}_{\rho}\;\nabla_{\dot{B}}\overline{W}^2\Big|\right) + \\
&+\frac{3}{16}i\alpha e\;W^2\big|\,\psi^{\mu\dot{B}}\psi_{\mu\dot{B}}\psi^A_{\nu}\;\nabla^{\dot{A}}\overline{W}^2\Big|\,\sigma^{\nu}_{A\dot{A}} + \frac{3}{16}i\alpha e\;\overline{W}^2\Big|\,\psi^{\mu B}\psi_{\mu B}\psi^{\dot{A}}_{\nu}\;\nabla^A W^2\big|\,\sigma^{\nu}_{A\dot{A}} + \\
&+\frac{3}{8}i\alpha e\psi^{\mu A}\psi^{\dot{A}}_{\mu}\sigma^{\nu}_{A\dot{A}}\left(W^2\big|\,\psi^{\dot{B}}_{\nu}\;\nabla_{\dot{B}}\overline{W}^2\Big| - \overline{W}^2\Big|\,\psi^B_{\nu}\;\nabla_B W^2\big|\right) - \\
&-\frac{3}{8}i\alpha e\sigma^{\nu}_{A\dot{A}}\left(W^2\big|\,\psi^{\mu A}\psi^{\dot{B}}_{\mu}\psi_{\nu\dot{B}}\;\nabla^{\dot{A}}\overline{W}^2\Big| + \overline{W}^2\Big|\,\psi^B_{\nu}\psi^{\mu}_{B}\psi^{\dot{A}}_{\mu}\;\nabla^A W^2\big|\right) - \\
&-\frac{\alpha}{3}e\left(M^2 + N^2 - A^\mu A_\mu\right)W^2\big|\;\overline{W}^2\Big| + \frac{\alpha}{4}e\;\nabla^A W^2\big|\;\nabla^{\dot{A}}\overline{W}^2\Big|\,\sigma^{\nu}_{A\dot{A}}A_\nu - \\
&-\frac{\alpha}{4}e\,(M + iN)\;\overline{W}^2\Big|\;\nabla^2 W^2\big| - \frac{\alpha}{4}e\,(M - iN)\;W^2\big|\;\overline{\nabla}^2\overline{W}^2\Big| + \\
&+i\alpha e\left(\overline{W}^2\Big|\,\partial_\mu\,W^2\big| - W^2\big|\,\partial_\mu\,\overline{W}^2\Big|\right)A^\mu - \\
&-\frac{i}{2}\alpha e\left(\overline{W}^2\Big|\;\nabla^A W^2\big|\,\psi^{\mu}_{A} - W^2\big|\;\nabla^{\dot{A}}\overline{W}^2\Big|\,\psi^{\mu}_{\dot{A}}\right)A_\mu\,.
\end{aligned}
\tag{2.3}
$$

In order to compute this lagrangian in terms of the x-space fields, we are interested in the component expansion of W^2. From (A.15), one can derive

$$
\begin{aligned}
W^2\big| = {}& \frac{1}{6}\psi^{mnC}\psi_{mnC} - \frac{i}{24}\varepsilon^{mnrs}\psi^C_{mn}\psi_{rsC} - \frac{i}{24}\varepsilon^{mnrt}\psi^A_{mn}\psi_{rsC}\left(\sigma_t{}^s\right)^C{}_A + \\
&+\frac{1}{12}A_nA^m\psi^{nA}\psi_{mA} - \frac{1}{12}A^mA_m\psi^{nA}\psi_{nA} - \\
&-\frac{i}{48}\varepsilon^{mnrt}A_sA_m\psi^A_n\psi_{rC}\left(\sigma_t{}^s\right)^C{}_A + \frac{i}{3}\psi^{mnA}A_m\psi_{nA} + \\
&+\frac{1}{12}\varepsilon^{mnrs}A_r\psi^C_{mn}\psi_{sC} + \frac{1}{12}\varepsilon^{mnrt}\psi^A_{mn}A_r\psi_{sC}\left(\sigma_t{}^s\right)^C{}_A
\end{aligned}
\tag{2.4}
$$

From the relation

$$
\nabla_A W_{BCD} = \nabla_{\underline{A}}W_{\underline{BCD}} - \frac{1}{4}\varepsilon_{AB}\nabla^E W_{ECD} - \frac{1}{4}\varepsilon_{AC}\nabla^E W_{EBD} - \frac{1}{4}\varepsilon_{AD}\nabla^E W_{EBC} \tag{2.5}
$$

one can easily show that one may write $\nabla_A W^2 = -2W^{BCD}\nabla_A W_{BCD}$ as

$$
\nabla_A W^2 = \frac{1}{2}W_A{}^{BC}\nabla^D W_{DBC} - 2W^{BCD}\nabla_{\underline{A}}W_{\underline{BCD}} \tag{2.6}
$$

and, using (A.5), one can also show that

$$
\begin{aligned}
\nabla^A W_{ABC}\big| &= -2i\sigma^{mn}_{BC}\;\nabla_m G_n\big| \\
&= -\frac{2}{3}i\sigma^{mn}_{BC}e_m{}^\mu\mathcal{D}_\mu A_m + i\sigma^{mn}_{BC}\psi_m{}^A\;\nabla_A G_n\big| + i\sigma^{mn}_{BC}\psi_m{}^{\dot{A}}\;\nabla_{\dot{A}}G_n\big|
\end{aligned}
\tag{2.7}
$$

In the same way one has, from (2.5) and the solution to the Bianchi identities,

$$\begin{aligned} \nabla^2 W^2 &= -2\left(\nabla^A W^{BCD}\right)\nabla_A W_{BCD} + 2W^{ABC}\nabla^2 W_{ABC} \\ &= -2\left(\nabla^A W^{BCD}\right)\nabla_{\underline{A}} W_{\underline{BCD}} + 12\left(\nabla^m G^n\right)\nabla_m G_n - \\ &\quad - 12\left(\nabla^m G^n\right)\nabla_n G_m - 12i\varepsilon^{mnrs}\left(\nabla_m G_n\right)\nabla_r G_s - \\ &\quad - \frac{5}{3}RW^2 - 20W^{ABC}G_A{}^{\dot{A}}\nabla_B G_{C\dot{A}} + 8iW^{ABC}\nabla_A{}^{\dot{A}}\nabla_B G_{C\dot{A}} \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} \nabla_{\underline{A}}{}^{\dot{A}}\nabla_{\underline{B}} G_{\underline{C}\dot{A}}\Big| &= \sigma_{\underline{A}}^{m\ \dot{A}} e_m{}^\mu \mathcal{D}_\mu\ \nabla_{\underline{B}} G_{\underline{C}\dot{A}}\Big| - 2\left(M - iN\right)\sigma_{\underline{AB}}^{mn}\psi_{m\underline{C}} A_n + \\ &\quad + \frac{i}{6}\sigma_{\underline{AB}}^{mn}\psi_{m\underline{C}} e_n{}^\mu \partial_\mu\left(M - iN\right) - \frac{i}{48}\sigma_{\underline{AB}}^{mn}\psi_{m\underline{C}}\psi_n^{\dot{D}}\ \nabla_{\dot{D}} R\big| + \\ &\quad + \frac{i}{2}\sigma_{\underline{AB}}^{mn}\sigma_{\underline{C}\dot{D}}^{p}\psi_p^{\dot{D}}\ \nabla_m G_n\big| - \frac{1}{2}\sigma_{\underline{A}}^{m\ \dot{A}}\psi_m^{\dot{D}}\ \nabla_{\underline{\dot{D}}}\nabla_{\underline{B}} G_{\underline{C\dot{A}}}\Big| \end{aligned} \tag{2.9}$$

We have thus expressed the x-space components necessary to compute our lagrangian in terms of the independent components listed in (A.10). These components, as well as $\nabla_m G_n|$, are presented in appendix A. Some of them are presented for the first time in the literature. The knowledge of these components allows us to fully compute the lagrangian (2.3), leaving the auxiliary fields in it.

In [1], we computed the field equations for (1.1). We showed that the auxiliary fields M, N satisfied an algebraic field equation, in terms of $e_\mu{}^m$, ψ_μ and the auxiliary field A_m. This field equation can be obtained by taking the $\theta = 0$ component of

$$R = 6\alpha\overline{W}^2\nabla^2 W^2 + 12\alpha^2\overline{W}^4 W^2\nabla^2 W^2 \tag{2.10}$$

These auxiliary fields can then be eliminated, leaving an action with higher powers of α and A_m.

The elimination of A_m is much more problematic. Expressing this auxiliary field in terms of $e_\mu{}^m$ and ψ_μ, we get a series of terms with infinite powers of α. This series cannot be expressed in a closed form; it is the solution of a differential equation - the field equation for A_m - the solution of which can only be iterated order by order in α. Ingenuously, one would expect the effective action (1.1) to be just of order κ^4 (or α'^3) but, from what we said, the closeness of the supersymmetry algebra requires, because of the elimination of A_m, the appearance of terms of infinitely high order in κ (or α'). Notice that these terms all come from the auxiliary field sector and do not include terms which are pure (i.e. without matter couplings) powers of the Riemann tensor. These pure Riemann terms obviously exist in the full loop (α') expansion, which is *not* fully given by (1.1). The full quantum action will include more terms as (1.1) but with larger powers of the Riemann tensor. Each of them will *presumably* need, on-shell, an infinite number of terms with all powers of the Riemann tensor to be supersymmetrized, like the simplest one we have analyzed.

The three-loop effective action (1.1) was written based on the dimensional analysis of its leading bosonic part and on the requirement of supersymmetry. Since any regularization procedure works order by order in perturbation theory, it cannot introduce (at three

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loops or at any other loop) an infinite number of terms of arbitrary order in the coupling constant, as it is required by supersymmetry. Therefore, it is not possible to fully preserve supersymmetry while regulating the theory; supersymmetry transformations must be truncated to the power of κ at which the theory is being regulated.

Having this in mind, now we compute the three-loop (or α'^3) terms in $\nabla^A W^2\big|$ and $\nabla^2 W^2\big|$, i.e. the terms which would not contribute to (2.3) with more powers of α. Since all these terms are already multiplied by α, the terms we are looking for are simply the $\alpha = 0$ (on-shell) terms of $\nabla_A W^2\big|$ and $\nabla^2 W^2\big|$, i.e. the terms without auxiliary fields. From (2.6), (2.8) and the components in appendix A, it is only a matter of calculation. We leave the details of the calculation to appendix B. Here, we present just the results:

$$
\begin{aligned}
\nabla_A W^2\big| = & -\frac{1}{3}\mathcal{W}^+_{\mu\nu\rho\sigma}\psi^{\mu\nu B}\sigma^{\rho\sigma}_{AB} + \frac{i}{24}\sigma^{mn}_{AB}\sigma^{sD\dot{A}}\psi^B_{rs}\psi_{mn\dot{A}}\psi^r_D + \frac{i}{12}\sigma^{ms}_{\underline{A}D}\sigma^r_{\underline{B}\dot{A}}\psi^n{}_s{}^B\psi^{\dot{A}}_{mn}\psi^D_r - \\
& - \frac{i}{4}\sigma^r_{\underline{B}\dot{A}}\psi^{mnB}\psi^{\dot{A}}_{mn}\psi_{r\underline{A}} - \frac{1}{8}\varepsilon^{mnrs}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rs}\psi^{\dot{A}}_{mn}\psi_{p\underline{A}} + \frac{1}{12}\varepsilon_{mnrs}\sigma^r_{\underline{B}\dot{A}}\psi^{psB}\psi_p{}^{m\dot{A}}\psi^n_{\underline{A}} - \\
& - \frac{1}{48}\varepsilon^{mnrs}\sigma^{pq}_{AB}\sigma_m{}^{D\dot{A}}\psi^B_{rs}\psi_{pq\dot{A}}\psi_{nD} + \frac{1}{24}\varepsilon^{mnrs}\sigma^q{}_{sAD}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rq}\psi^{\dot{A}}_{mn}\psi^D_p + \\
& + \frac{1}{24}\varepsilon^{mnrs}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rp}\psi^{\dot{A}}_{mn}\psi_{s\underline{A}} - \frac{1}{24}\varepsilon^{mnrs}\sigma_{s\underline{B}\dot{A}}\psi^B_{rp}\psi^{\dot{A}}_{mn}\psi^p_{\underline{A}} + \\
& + \frac{i}{24}\sigma^{mn}_{AB}\sigma^{sD\dot{A}}\psi^B_{rs}\psi^r_{\dot{A}}\psi_{mnD} + \frac{i}{12}\sigma^{ms}_{\underline{A}D}\sigma^r_{\underline{B}\dot{A}}\psi^n{}_s{}^B\psi^{\dot{A}}_r\psi^D_{mn} - \frac{i}{4}\sigma^r_{\underline{B}\dot{A}}\psi^B_{mn}\psi^{\dot{A}}_r\psi^{mn}_{\underline{A}} - \\
& - \frac{1}{8}\varepsilon^{mnrs}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rs}\psi^{\dot{A}}_p\psi_{mn\underline{A}} + \frac{1}{12}\varepsilon_{mnrs}\sigma^r_{\underline{B}\dot{A}}\psi^{psB}\psi^{n\dot{A}}\psi_p{}^m{}_{\underline{A}} - \\
& - \frac{1}{48}\varepsilon^{mnrs}\sigma^{pq}_{AB}\sigma_m{}^{D\dot{A}}\psi^B_{rs}\psi_{n\dot{A}}\psi_{pqD} + \frac{1}{24}\varepsilon^{mnrs}\sigma^q{}_{sAD}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rq}\psi^{\dot{A}}_p\psi^D_{mn} + \\
& + \frac{1}{24}\varepsilon^{mnrs}\sigma^p_{\underline{B}\dot{A}}\psi^B_{rp}\psi^{\dot{A}}_s\psi_{mn\underline{A}} - \frac{1}{24}\varepsilon^{mnrs}\sigma_{s\underline{B}\dot{A}}\psi^B_{rp}\psi^{p\dot{A}}\psi_{mn\underline{A}} + \mathcal{O}(\alpha)
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
\nabla^2 W^2\big| = & -2\mathcal{W}^+_{\mu\nu\rho\sigma}\mathcal{W}_+^{\mu\nu\rho\sigma} + \frac{8}{3}i\mathcal{W}^+_{\mu\nu\rho\sigma}\psi^{\rho A}\psi^{\mu\nu\dot{A}}\sigma^\sigma_{A\dot{A}} - \frac{8}{3}i\mathcal{W}^+_{\mu\nu\rho\sigma}\psi^{\mu\nu A}\psi^{\rho\dot{A}}\sigma^\sigma_{A\dot{A}} + \\
& + \frac{1}{8}\sigma^{mn}_{\underline{AB}}\sigma^r_{\underline{C}}{}^{\dot{A}}\sigma^{pqAB}\sigma^{sC\dot{B}}\psi_{r\underline{D}}\psi_{mn\dot{A}}\psi^D_s\psi_{pq\dot{B}} + \\
& + \frac{1}{8}\sigma^{mn}_{\underline{AB}}\sigma^r_{\underline{C}}{}^{\dot{A}}\sigma^{pqAB}\sigma^{sC\dot{B}}\psi_{mn\underline{D}}\psi_{r\dot{A}}\psi^D_{pq}\psi_{s\dot{B}} - \\
& - \frac{1}{4}\sigma^{mn}_{\underline{AB}}\sigma^r_{\underline{C}}{}^{\dot{A}}\sigma^{pqAB}\sigma^{sC\dot{B}}\psi_{r\underline{D}}\psi_{mn\dot{A}}\psi^D_{pq}\psi_{s\dot{B}} + \mathcal{O}(\alpha)
\end{aligned}
\tag{2.12}
$$

The product $\sigma^{mn}_{AB}\sigma^r_C{}^{\dot{A}}\sigma^{pq\underline{AB}}\sigma^{s\underline{C}\dot{B}}\varepsilon^{\underline{DE}}$ is expanded in appendix B.

When replaced in (2.3), these expressions will give all (and no more than that) terms which contribute with a power of κ^4 (or α'^3). This concludes our calculation of the three-loop $\mathcal{N} = 1, d = 4$ supergravity effective action, except for the numerical factor in the definition of α.

3. List and discussion of the different $\mathcal{R}^4$ superinvariants

As proven in [7], in four dimensions there are thirteen independent real scalar polynomials made from four powers of the irreducible components of the Riemann tensor, besides the

$\mathcal{W}_+^2\mathcal{W}_-^2$ which we analyzed in the previous section.[1] These terms are proportional to $\mathcal{R}$ or $\mathcal{S}_{\mu\nu}$ and therefore, when written in superspace language, they are proportional to R or G_m, with the single exception of $\mathcal{W}_+^4+\mathcal{W}_-^4$, which we analyze at the end.

For each fourth-order scalar polynomial of the Riemann tensor we now give its superspace supersymmetric form, when it exists. For all the component expansions see appendix A. Also from these component expansions, or from the off-shell relations (A.4), (A.5), (A.6), we can derive the R-weights of the superfields and their derivatives [8]:

$$\nabla_A \mapsto +1\,, \qquad R \mapsto +2\,, \qquad G_m \mapsto 0\,, \qquad W_{ABC} \mapsto -1 \tag{3.1}$$

Having these weights, we can mention in advance that all the actions we are going to consider preserve R-symmetry.

3.1 $\mathcal{R}^4$

The supersymmetrization of the $\mathcal{R}^4$ polynomial is written, in superspace, as

$$\kappa^4 \int ER\overline{R}\left(\overline{\nabla}^2 R\right)\left(\nabla^2\overline{R}\right)d^4\theta = -\frac{3}{2}\kappa^4\int \epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[R\overline{R}\left(\overline{\nabla}^2 R\right)\left(\nabla^2\overline{R}\right)\right]d^2\theta + \text{h.c.} \tag{3.2}$$

This lagrangian contains a $\frac{3}{8}\kappa^4 e\left(\nabla^2\overline{R}\right)^2\Big|\left(\overline{\nabla}^2 R\right)^2\Big|$ term, the bosonic part of which being $1536\kappa^4 e\mathcal{R}^4$.

3.2 $\mathcal{R}^2\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}$

The supersymmetrization of the $\mathcal{R}^2\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}$ polynomial is included in the following superspace lagrangian:

$$\kappa^4 \int EG^2\left(\overline{\nabla}^2 R\right)\left(\nabla^2\overline{R}\right)d^4\theta = -\frac{3}{2}\kappa^4\int \epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[G^2\left(\overline{\nabla}^2 R\right)\left(\nabla^2\overline{R}\right)\right]d^2\theta + \text{h.c.} \tag{3.3}$$

This lagrangian contains a $\frac{3}{16}\kappa^4 e\, \nabla^2\overline{R}\big|\,\overline{\nabla}^2 R\big|\left(\overline{\nabla}^2\nabla^2+\nabla^2\overline{\nabla}^2\right)G^2\Big|$ term. From (A.33), and since $\nabla^{\underline{A}}\nabla^{\underline{\dot{A}}}G^{\underline{B\dot{B}}}\Big|\,\nabla_A\nabla_{\dot{A}}G_{B\dot{B}}\big| = \mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}+$fermion terms, $\nabla^2\overline{\nabla}^2 G^2\Big| = -\frac{1}{18}\mathcal{R}^2-2\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}+$ fermion terms, the bosonic part of this lagrangian is given by $-\frac{4}{3}\kappa^4 e\mathcal{R}^4-48\kappa^4 e\mathcal{R}^2\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}$.

3.3 $\mathcal{R}\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\nu\sigma}$

The supersymmetrization of the $\mathcal{R}\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\nu\sigma}$ polynomial is written, in superspace, as

$$\begin{aligned}
&\kappa^4\int E\left[R\left(\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\right)\left(\nabla^{\dot{C}}G^{A\dot{A}}\right)\left(\nabla_{\dot{C}}G^{B\dot{B}}\right)+\text{h.c.}\right]d^4\theta \qquad (3.4)\\
&\quad = -\frac{3}{2}\kappa^4\int\epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[R\left(\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\right)\left(\nabla^{\dot{C}}G^{A\dot{A}}\right)\left(\nabla_{\dot{C}}G^{B\dot{B}}\right)+\text{h.c.}\right]d^2\theta+\text{h.c.}
\end{aligned}$$

[1] We recall that the superfield components which contain the Riemann tensor are $\nabla^2\overline{R}\big|$ containing $\mathcal{R}$, $\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big|$ containing $\mathcal{S}_{\mu\nu}=\mathcal{R}_{\mu\nu}-\frac{1}{4}\mathcal{R}g_{\mu\nu}$, and $\nabla_{\underline{A}}W_{\underline{BCD}}\big|$ containing $\mathcal{W}_{ABCD} := -\frac{1}{8}\mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}$ (see appendix A).

This action contains a $\frac{3}{16}\kappa^4 e\ \overline{\nabla}^2 R\Big|\ \nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big|\ \nabla^C\nabla^{\dot{C}}G^{A\dot{A}}\Big|\ \nabla_C\nabla_{\dot{C}}G^{B\dot{B}}\Big|$+h.c. term. Since $\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big|\ \nabla^C\nabla^{\dot{C}}G^{A\dot{A}}\Big|\ \nabla_C\nabla_{\dot{C}}G^{B\dot{B}}\Big| = -\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\nu\sigma}$ + fermion terms, the bosonic part of this action is indeed given by $3\kappa^4 e\mathcal{R}\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\nu\sigma}$.

3.4 $\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}\mathcal{S}_{\rho\sigma}\mathcal{S}^{\rho\sigma}$

The supersymmetrization of the $\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}\mathcal{S}_{\rho\sigma}\mathcal{S}^{\rho\sigma}$ polynomial is included in the following superspace lagrangian:

$$\kappa^4\int E\left(\overline{\nabla}^2 G^2\right)\nabla^2 G^2 d^4\theta \tag{3.5}$$

This lagrangian contains a $\frac{3}{8}\kappa^4 e\ \nabla^2\overline{\nabla}^2 G^2\Big|\ \overline{\nabla}^2\nabla^2 G^2\Big|$ term. Again from (A.33), the bosonic part of this lagrangian is $\frac{3}{2}\kappa^4 e\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu}\mathcal{S}_{\rho\sigma}\mathcal{S}^{\rho\sigma} + \frac{1}{12}\kappa^4 e\mathcal{R}^2\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu\nu} + \frac{1}{432}\kappa^4 e\mathcal{R}^4$.

3.5 $\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\rho\sigma}\mathcal{S}^{\nu}{}_{\rho}$

The supersymmetrization of the $\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\rho\sigma}\mathcal{S}^{\nu}{}_{\rho}$ polynomial is written, in superspace, as

$$\begin{aligned}\kappa^4 &\int E\left(\nabla^{\dot{C}}G^{A\dot{A}}\right)\left(\nabla_{\dot{C}}G^{B\dot{B}}\right)\left(\nabla^C G_{A\dot{A}}\right)\left(\nabla_C G^{B\dot{B}}\right)d^4\theta \\ &= -\frac{3}{2}\kappa^4\int\epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[\left(\nabla^{\dot{C}}G^{A\dot{A}}\right)\left(\nabla_{\dot{C}}G^{B\dot{B}}\right)\left(\nabla^C G_{A\dot{A}}\right)\left(\nabla_C G^{B\dot{B}}\right)\right]d^2\theta + \text{h.c.}\end{aligned} \tag{3.6}$$

This action contains a $\frac{3}{8}\kappa^4 e\ \nabla^{\underline{A}}\nabla^{\underline{\dot{A}}}G^{\underline{C\dot{C}}}\Big|\ \nabla_A\nabla_{\dot{A}}G^{D\dot{D}}\Big|\ \nabla^{\dot{B}}\nabla^B G_{C\dot{C}}\Big|\ \nabla_{\underline{\dot{B}}}\nabla_{\underline{B}}G_{\underline{D\dot{D}}}\Big|$ term, the bosonic part of which being given by $\frac{3}{8}\kappa^4 e\mathcal{S}_{\mu\nu}\mathcal{S}^{\mu}{}_{\sigma}\mathcal{S}^{\rho\sigma}\mathcal{S}^{\nu}{}_{\rho}$.

3.6 $\mathcal{R}\mathcal{S}_{\mu\rho}\mathcal{S}_{\nu\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$

The supersymmetrization of the $\mathcal{R}\mathcal{S}_{\mu\rho}\mathcal{S}_{\nu\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ polynomial is written, in superspace, as

$$\begin{aligned}\kappa^4 &\int E\left[\overline{R}\left(\nabla^A G^{B\dot{B}}\right)\left(\nabla^C G^D{}_{\dot{B}}\right)\nabla_D W_{ABC} - R\left(\nabla^{\dot{A}}G^{B\dot{B}}\right)\left(\nabla^{\dot{C}}G_B{}^{\dot{D}}\right)\nabla_{\dot{D}}W_{\dot{A}\dot{B}\dot{C}}\right]d^4\theta \\ &= -\frac{3}{2}\kappa^4\int\epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[\overline{R}\left(\nabla^A G^{B\dot{B}}\right)\left(\nabla^C G^D{}_{\dot{B}}\right)\nabla_D W_{ABC} + \text{h.c.}\right]d^2\theta + \text{h.c.}\end{aligned} \tag{3.7}$$

This lagrangian contains a $\frac{3}{16}\kappa^4 e\ \nabla^2\overline{R}\big|\ \nabla^{\dot{A}}\nabla^A G^{B\dot{B}}\Big|\ \nabla_{\underline{\dot{A}}}\nabla^C G^D{}_{\underline{\dot{B}}}\Big|\ \nabla_{\underline{A}}W_{\underline{BCD}}\big|$ + h.c. term. Using (B.3) one can derive $\nabla^{\dot{A}}\nabla^A G^{B\dot{B}}\Big|\ \nabla_{\underline{\dot{A}}}\nabla^C G^D{}_{\underline{\dot{B}}}\Big|\ \nabla_{\underline{A}}W_{\underline{BCD}}\big| = -\frac{4}{3}\mathcal{S}_{\mu\rho}\mathcal{S}_{\nu\sigma}\mathcal{W}_+^{\mu\nu\rho\sigma}$ + fermionic terms, and one can then show that the pure bosonic part of this lagrangian is $2\kappa^4 e\mathcal{R}\mathcal{S}_{\mu\rho}\mathcal{S}_{\nu\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$.

3.7 $\mathcal{R}^2\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$

The supersymmetrization of the $\mathcal{R}^2\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ polynomial is written, in superspace, as

$$\kappa^4\int ER\overline{R}\left(\nabla^2 W^2+\overline{\nabla}^2\overline{W}^2\right)d^4\theta = -\frac{3}{2}\kappa^4\int\epsilon\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\left[R\overline{R}\left(\overline{\nabla}^2\overline{W}^2+\nabla^2 W^2\right)\right]d^2\theta + \text{h.c.} \tag{3.8}$$

This lagrangian contains a $\frac{3}{8}\kappa^4 e\ \nabla^2\overline{R}\big|\ \overline{\nabla}^2 R\Big|\ \left(\nabla^2 W^2+\overline{\nabla}^2\overline{W}^2\right)\Big|$ term, the bosonic part of which being given by $-48\kappa^4 e\mathcal{R}^2\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$.

3.8 $\mathcal{S}_{\tau\lambda}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$

The supersymmetrization of the $\mathcal{S}_{\tau\lambda}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ polynomial is included in the following superspace lagrangian:

$$\kappa^4 \int EG^2\left(\nabla^2 W^2 + \overline{\nabla}^2\overline{W}^2\right) d^4\theta$$
$$= -\frac{3}{2}\kappa^4 \int \epsilon\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left[G^2\left(\overline{\nabla}^2\overline{W}^2 + \nabla^2 W^2\right)\right] d^2\theta + \text{h.c.} \tag{3.9}$$

This lagrangian contains a $\frac{3}{16}\kappa^4 e\left(\overline{\nabla}^2\nabla^2 + \nabla^2\overline{\nabla}^2\right)G^2\Big|\left(\nabla^2 W^2 + \overline{\nabla}^2\overline{W}^2\right)\Big|$ term. Its pure bosonic part is given by $\frac{1}{24}\kappa^4 e\mathcal{R}^2\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma} + \frac{3}{2}e\kappa^4\mathcal{S}_{\tau\lambda}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$.

3.9 $\mathcal{S}_{\mu\nu}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\tau\rho\lambda\sigma}\mathcal{W}^{\mu\rho\nu\sigma}$

The supersymmetrization of the $\mathcal{S}_{\mu\nu}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\tau\rho\lambda\sigma}\mathcal{W}^{\mu\rho\nu\sigma}$ polynomial contains the following superspace lagrangian:

$$\kappa^4 \int EW_{BCD}W_{\dot{B}\dot{C}\dot{D}}\left(\nabla^{\dot{B}}G^{B\dot{C}}\right)\nabla^C G^{D\dot{D}} d^4\theta$$
$$= -\frac{3}{2}\kappa^4 \int \epsilon\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left(W_{BCD}W_{\dot{B}\dot{C}\dot{D}}\left(\nabla^{\dot{B}}G^{B\dot{C}}\right)\nabla^C G^{D\dot{D}}\right) d^2\theta + \text{h.c.} \tag{3.10}$$

This lagrangian contains a $\frac{3}{8}\kappa^4 e\, \nabla_{\underline{A}}W_{\underline{BCD}}\big|\ \nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\Big|\ \nabla^A\nabla^{\dot{B}}G^{B\dot{C}}\Big|\ \nabla^{\dot{A}}\nabla^C G^{D\dot{D}}\Big|$ term. Since $\nabla_{\underline{A}}W_{\underline{BCD}}\big|\ \nabla^A\nabla_{\underline{\dot{A}}}G^B{}_{\underline{\dot{B}}}\Big| = \frac{1}{12}\mathcal{S}^\mu{}_\tau\mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}_{CD}\sigma^{\tau\nu}_{\dot{A}\dot{B}} + \frac{2}{3}\mathcal{W}^+_{\mu\nu\rho\sigma}\mathcal{S}^{\mu\rho}\sigma^\nu_{\underline{C\dot{A}}}\sigma^\sigma_{\underline{D\dot{B}}}$ + fermionic terms, using the symmetries of the Weyl tensor one can derive the pure bosonic part of this lagrangian, which is $-\frac{4}{3}\kappa^4 e\mathcal{S}_{\mu\nu}\mathcal{S}^{\tau\lambda}\mathcal{W}^+_{\tau\rho\lambda\sigma}\mathcal{W}^{\mu\rho\nu\sigma}_-$.

3.10 $\mathcal{S}_\mu{}^\tau\mathcal{S}_\nu{}^\lambda\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$

The supersymmetrization of the $\mathcal{S}_\mu{}^\tau\mathcal{S}_\nu{}^\lambda\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ polynomial is written, in superspace, as

$$\kappa^4 \int E\left[W_{ABC}W^{CEF}\left(\nabla^A G_{E\dot{B}}\right)\nabla^B G_F{}^{\dot{B}} + \text{h.c.}\right] d^4\theta$$
$$= -\frac{3}{2}\kappa^4 \int \epsilon\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left[W_{ABC}W^{CEF}\left(\nabla^A G_{E\dot{B}}\right)\nabla^B G_F{}^{\dot{B}} + \text{h.c.}\right] d^2\theta + \text{h.c.} \tag{3.11}$$

This lagrangian contains a $-\frac{3}{16}\kappa^4 e\, \nabla_{\underline{A}}W_{\underline{BCD}}\big|\ \nabla^C W^{DEF}\big|\ \nabla^{\underline{\dot{A}}}\nabla^A G_{E\underline{\dot{B}}}\Big|\ \nabla_{\dot{A}}\nabla^B G_F{}^{\dot{B}}\Big| + \text{h.c.}$ term. Since $\mathcal{W}^{CDBF}\mathcal{W}_{CDAE} = \frac{1}{8}\sigma^{\mu\nu}_{BF}\sigma^{AE}_{\tau\lambda}\mathcal{W}^+_{\mu\nu\rho\sigma}\mathcal{W}^{\tau\lambda\rho\sigma}_+$, the pure bosonic part of this lagrangian is easily shown to be $\frac{3}{8}\kappa^4 e\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}\mathcal{S}_\mu{}^\tau\mathcal{S}_\nu{}^\lambda$.

3.11 $\mathcal{R}\mathcal{W}_{\mu\nu}{}^{\tau\lambda}\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$

The supersymmetrization of the $\mathcal{R}\mathcal{W}_{\mu\nu}{}^{\tau\lambda}\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ polynomial is written, in superspace, as

$$\kappa^4 \int E\left(RW^{ABC}W_A{}^{DE}\nabla_B W_{CDE} + \text{h.c.}\right) d^4\theta$$
$$= -\frac{3}{2}\kappa^4 \int \epsilon\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left(RW^{ABC}W_A{}^{DE}\nabla_B W_{CDE} + \text{h.c.}\right) d^2\theta + \text{h.c.} \tag{3.12}$$

This lagrangian contains a $\frac{3}{16}\kappa^4 e\, \overline{\nabla}^2 R\Big|\, \nabla^F W^{ABC}\big|\, \nabla_F W_A{}^{DE}\big|\, \nabla_B W_{CDE}\big| + $h.c. term. Since $\mathcal{W}^{ABCF}\mathcal{W}_{AF}{}^{DE}\mathcal{W}_{BCDE} = -\frac{4}{9}\mathcal{W}^{+}_{\mu\nu}{}^{\tau\lambda}\mathcal{W}^{+}_{\tau\lambda\rho\sigma}\mathcal{W}_{+}^{\mu\nu\rho\sigma}$, the pure bosonic part of this lagrangian is $\frac{2}{3}\kappa^4 e\mathcal{R}\mathcal{W}_{\mu\nu}{}^{\tau\lambda}\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$.

3.12 Discussion of the results

We have identified, for a total of twelve independent real scalar polynomials made from four powers of irreducible components of the Riemann tensor, their corresponding superspace lagrangians. Obviously our choice of basis for these polynomials is not unique, and linear combinations of the superspace lagrangians we found can be taken freely, in order to supersymmetrize any desired linear combination of the Riemann polynomials.

Also other contractions of indices could have been taken, both in the Riemann polynomials and in the superspace lagrangians, but they would always be equivalent to some linear combination of the independent lagrangians we chose. For example, let's consider

$$\begin{aligned}\kappa^4 \int E\left[W_{BCD}W^{BEF}\left(\nabla_E G_{F\dot{B}}\right)\nabla^C G^{D\dot{B}} + \text{h.c.}\right] d^4\theta \qquad (3.13)\\ = -\frac{3}{2}\kappa^4 \int \epsilon\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\left[W_{BCD}W^{BEF}\left(\nabla_E G_{F\dot{B}}\right)\nabla^C G^{D\dot{B}} + \text{h.c.}\right] d^2\theta + \text{h.c.}\end{aligned}$$

This lagrangian contains a $\frac{3}{16}\kappa^4 e\, \nabla_{\underline{A}} W_{\underline{BCD}}\big|\, \nabla^A W^{BEF}\big|\, \nabla_{\underline{\dot{A}}}\nabla_E G_{F\underline{\dot{B}}}\Big|\, \nabla^{\dot{A}}\nabla^C G^{D\dot{B}}\Big| + $ h.c. term. From our previous results, it is easy to show that its pure bosonic part is given by $\frac{1}{24}\kappa^4 e\mathcal{S}_\mu{}^\tau\mathcal{S}_\nu{}^\lambda\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma} + \frac{1}{96}\kappa^4 e\mathcal{S}_{\tau\lambda}\mathcal{S}^{\tau\lambda}\mathcal{W}_{\mu\nu\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$. It is easy to see that it is a linear combination of the bosonic parts of (3.8), (3.9) and (3.11). Indeed, by partial integration in superspace, one can see directly that (3.13) is a linear combination of (3.8), (3.9) and (3.11).

All the supersymmetrizations we have been considering, as we mentioned, are proportional to R or G_m. If they are written simply as quantum corrections to pure supergravity, they can be reabsorbed by field redefinitions, according to [2, 9]. In superspace this can be seen very clearly. We recall that the general variation of the supergravity action $I = \frac{1}{2\kappa^2}\int\int Ed^4\theta d^4x$ under any transformation of the supervielbein which preserves the off-shell torsion constraints is given by [10]

$$2\kappa^2\delta I = \int E\left(\frac{2}{3}i\chi^{A\dot{A}}G_{A\dot{A}} + \frac{1}{9}\overline{RU} + \frac{1}{9}RU\right) d^4x d^4\theta \qquad (3.14)$$

The superfields $U, \chi^{A\dot{A}}$ are completely arbitrary. Any correction term proportional to R or G_m can be written in the form (3.14) and, therefore, reabsorbed in the supergravity action by a redefinition of the supervielbein. In fact, these correction terms to pure supergravity vanish on shell, because the field equations for pure supergravity are $R = 0$ and $G_m = 0$. The supervielbein redefinition will, though, introduce new higher order superinvariants (by higher order, in this section, we mean κ^{10} or more), which may vanish on-shell or not, as we will see below.

The correction in (1.1) cannot be written in the form (3.14), nor can its variation. In this case, the variation is much more complicated and, although still being proportional to the arbitrary superfields $U, \chi^{A\dot{A}}$, it includes terms with derivatives of R, G_m and W_{ABC} [1]. These terms imply the nontrivial field equations for the auxiliary fields.

There is an interesting aspect about the $\mathcal{R}\mathcal{W}_{\mu\nu}{}^{\tau\lambda}\mathcal{W}_{\tau\lambda\rho\sigma}\mathcal{W}^{\mu\nu\rho\sigma}$ action (3.12): it has three powers of the Weyl tensor, while all the other actions we have been considering in this section have at most two powers of the Weyl tensor. This means all the previous actions have, in superspace, up to derivatives, at least either two G_m factors, or one $R\overline{R}$ factor, or one $(R+\overline{R})G_m$ factor. It is then obvious that these actions will not change the supergravity field equations $R=0, G_m=0$ by themselves; any arbitrary variation of R or G_m, no matter how complicated it is, will always be multiplied by R or G_m (up to derivatives and complex conjugation) and, therefore, will not change any solution to the field equations. But that is not the case in the action (3.12): a variation of R would be multiplied by W_{ABC} terms which do not vanish on-shell and, if it is nontrivial, it could induce changes in the field equations.

Indeed, $R = -3iT_{A\dot{A}}{}^{A\dot{A}}$ and, in the notation of our previous paper [1],

$$\begin{aligned}\delta T_{A\dot{A}}{}^{A\dot{A}} = & -\frac{1}{2}H_{A\dot{A}}{}^{B\dot{B}}T_{B\dot{B}}{}^{A\dot{A}} + H^{AB}T_{A\dot{A}B}{}^{\dot{A}} + H^{A\dot{B}}T_{A\dot{A}\dot{B}}{}^{\dot{A}} + \frac{1}{2}H^{AB\dot{B}}T_{A\dot{A}B\dot{B}}{}^{\dot{A}} + \\ & + T_{A\dot{A}}{}^{AB}H_{B}{}^{\dot{A}} + T_{A\dot{A}}{}^{A\dot{B}}H_{\dot{B}}{}^{\dot{A}} - \nabla_{A\dot{A}}H^{A\dot{A}} + \nabla^{A}H_{A\dot{A}}{}^{\dot{A}} \end{aligned} \quad (3.15)$$

The torsion terms are all proportional to R or G_m and vanish on-shell. The same is true for the $\nabla_{A\dot{A}}H^{A\dot{A}}$ term (recall that the full set of H_M^N was computed in [1]). The $\nabla^A H_{A\dot{A}}{}^{\dot{A}}$ term introduces the following "dangerous" terms:

$$\delta R = \nabla^2\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} - \frac{3}{4}\nabla_{A\dot{A}}\chi^{A\dot{A}} - 3i\left(\nabla_A\nabla_{\dot{A}} - \nabla_{\dot{A}}\nabla_A\right)\chi^{A\dot{A}}\right] + \ldots \quad (3.16)$$

These terms are nonzero on-shell, if (3.12) is taken as a correction to pure supergravity, and will induce a change in the field equations. This action has the combined features of all the distinct actions we have analyzed: it changes the supergravity field equations like (1.1) but, because it is proportional to R, it can still be reabsorbed in the supergravity action, by a supervielbein redefinition, like all the actions we have analyzed in this section. Furthermore, because the supervielbein variations H_M^N include derivatives of χ^m and U, we get, after partial integration, terms with derivatives of R and G_m in the superspace field equations, which may mean that also with this action the auxiliary fields have nontrivial field equations.

After reabsorbing this term, both in pure supergravity and in (1.1), the new higher-order superinvariants which are generated are just powers of the Weyl tensor, which do not vanish on shell. Therefore, although this counterterm vanishes on-shell, it generates non-trivial higher-order superinvariants, even in pure supergravity (without the $\mathcal{W}^4$ correction in (1.1)).

All the other terms we have considered in this section up to now introduce higher powers of either $\mathcal{R}$ or $\mathcal{R}_{\mu\nu}$, which vanish on-shell if they are the only quantum corrections to supergravity. They will not vanish on-shell when we include the $\mathcal{W}^4$ correction in (1.1).

We emphasize again that the full loop (α') expansion is *not* fully given by (1.1), even after having the higher order terms from the redefinition of the supervielbein. These terms may need to be included in the full action with different numerical coefficients, and also other higher order terms, not obtained from the redefinition of the supervielbein, may be needed.

3.13 $\mathcal{W}^4$ terms

To conclude our analysis of the $\mathcal{R}^4_{\mu\nu\rho\sigma}$ superinvariants in four dimensions, we discuss now pure Weyl terms.

There are two possible $\mathcal{W}^4$ polynomials in four dimensions: the most interesting $\mathcal{W}^2_+\mathcal{W}^2_-$, which we analyzed in [1] and in section 2, and $\mathcal{W}^4_+ + \mathcal{W}^4_-$. This last term simply cannot be supersymmetrized, as noticed in [4]. Indeed, in components, this term would be written as $\left(\nabla^2 W^2\right|\right)^2 + \text{h.c.}$, which is not a supersymmetric combination (it cannot result from a superspace integration).

4. Conclusions

From the thirteen independent polynomials that can be built from $\mathcal{R}^4_{\mu\nu\rho\sigma}$, only twelve of them can be supersymmetrized and, from these twelve, only one — $\mathcal{W}^2_+\mathcal{W}^2_-$ — cannot be reabsorbed in the pure supergravity action by a redefinition of the supervielbein. If this term is included, the field equations are changed and one gets a nonlocal action after elimination of the auxiliary fields.

In this paper, we took the $\alpha\mathcal{W}^2_+\mathcal{W}^2_-$ case which we had already studied in superspace in a previous paper, and we worked it out in x-space. We wrote down the action in terms of the components of the W^2 superfield. We computed these components, leaving the auxiliary fields which, on-shell, are an infinite series in the coupling constant α. We worked these components out completely, just in terms of the vielbein and the gravitino, to zeroth order in α. We got then the supersymmetric $\mathcal{W}^2_+\mathcal{W}^2_-$ action, but the field equations of the auxiliary fields had to be truncated. This should be the three-loop supergravity effective action, assuming this theory is not finite to this order.

We showed that every other $\mathcal{R}^4_{\mu\nu\rho\sigma}$ term which has a $\mathcal{R}$ or $\mathcal{R}_{\mu\nu}$ factor is supersymmetrizable, but can be reabsorbed in the pure supergravity action by a redefinition of the supervielbein. For each of these terms, we wrote in superspace the respective supersymmetric completion and we proved its R-invariance. After the supervielbein redefinition, new higher powers of the Riemann tensor are generated, which we have not analyzed in detail; we argued, though, that among these terms should exist powers of the Weyl tensor.

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A. From superspace to components in $\mathcal{N}=1, d=4$ supergravity

Our conventions have been mostly described in [1]. The Riemann tensor admits, in d spacetime dimensions, the following decomposition in terms of the Weyl tensor $\mathcal{W}_{\mu\nu\rho\sigma}$, the Ricci tensor $\mathcal{R}_{\mu\nu}$ and the Ricci scalar $\mathcal{R}$:

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= \mathcal{W}_{\mu\nu\rho\sigma} - \frac{1}{d-2}\left(g_{\mu\rho}\mathcal{R}_{\nu\sigma} - g_{\nu\rho}\mathcal{R}_{\mu\sigma} + g_{\nu\sigma}\mathcal{R}_{\mu\rho} - g_{\mu\sigma}\mathcal{R}_{\nu\rho}\right) + \\ &+ \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)\mathcal{R}\end{aligned} \tag{A.1}$$

We define the traceless Ricci tensor as

$$\mathcal{S}_{\mu\nu} := \mathcal{R}_{\mu\nu} - \frac{1}{d}g_{\mu\nu}\mathcal{R} \tag{A.2}$$

In four dimensions, the Weyl tensor can still be decomposed in its self-dual and antiself-dual parts:

$$\mathcal{W}_{\mu\nu\rho\sigma} = \mathcal{W}^{+}_{\mu\nu\rho\sigma} + \mathcal{W}^{-}_{\mu\nu\rho\sigma}, \mathcal{W}^{\mp}_{\mu\nu\rho\sigma} := \frac{1}{2}\left(\mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2}\varepsilon_{\mu\nu}{}^{\lambda\tau}\mathcal{W}_{\lambda\tau\rho\sigma}\right) \tag{A.3}$$

Superspace supergravity is described by two antichiral superfields $\overline{R}, W_{ABC}$, their complex conjugates and a real superfield $G_{A\dot{A}}$, which describe the off-shell solution to the Bianchi identities. These identities imply the following differential relations between the superfields:

$$\nabla^A G_{A\dot{B}} = \frac{1}{24}\nabla_{\dot{B}}R \tag{A.4}$$

$$\nabla^A W_{ABC} = i\left(\nabla_{B\dot{A}}G_C{}^{\dot{A}} + \nabla_{C\dot{A}}G_B{}^{\dot{A}}\right) \tag{A.5}$$

From (A.4) and its complex conjugate and the solution of the Bianchi identities, we may also derive the following useful relation between superfields:

$$\nabla^2\overline{R} - \overline{\nabla}^2 R = 96i\nabla^n G_n \tag{A.6}$$

These relations (A.4), (A.5), (A.6) are off-shell identities (not field equations).

In order to determine the component expansion of the supergravity superfields, we use the method of gauge completion [11]–[14]. The basic idea behind it is to relate in superspace some superfields and superparameters at $\theta=0$ with some x space quantities, and then to require compatibility between the x space and superspace transformation rules.

We make the following identification for the supervielbeins at $\theta=0$ (symbolically $E_{\Pi}^{N}|$):

$$E_{\Pi}{}^{N}\big| = \begin{bmatrix} e_\mu{}^m & \frac{1}{2}\psi_\mu{}^A & \frac{1}{2}\psi_\mu{}^{\dot{A}} \\ 0 & \delta_B{}^A & 0 \\ 0 & 0 & \delta_{\dot{B}}{}^{\dot{A}} \end{bmatrix} \tag{A.7}$$

In the same way, we gauge the fermionic superconnection at order $\theta=0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\begin{aligned}\Omega_{\mu m}{}^{n}\big| &= \omega_{\mu m}{}^{n}(x) \\ \Omega_{Am}{}^{n}\big|, \Omega_{\dot{A}m}{}^{n}\big| &= 0\end{aligned} \tag{A.8}$$

We also identify, at the same order $\theta = 0$, the superspace vector covariant derivative (with an Einstein indice) with the curved space covariant derivative:

$$\nabla_\mu| = \mathcal{D}_\mu \tag{A.9}$$

These gauge choices are all preserved by supergravity transformations.

As a careful analysis using the solution to the Bianchi identities and the off-shell relations among the supergravity superfields $\overline{R}, G_n, W_{ABC}$ shows, the component field content of these superfields is all known once we know[2]

$$\overline{R}\big|\,, \quad \nabla_A\overline{R}\big|\,, \quad \nabla^2\overline{R}\big|\,, \quad G_{A\dot{A}}\big|\,, \quad \nabla_{\underline{A}}G_{\underline{B}\dot{A}}\Big|\,, \quad \nabla_{\underline{\dot{A}}}\nabla_{\underline{A}}G_{\underline{B}\underline{\dot{B}}}\Big|\,, \quad W_{ABC}\big|\,, \quad \nabla_{\underline{D}}W_{\underline{ABC}}\big| \tag{A.10}$$

and their complex conjugates.

All the other components and higher derivatives of $\overline{R}, G_{A\dot{A}}, W_{ABC}$ can be written as functions of these previous ones. In order to determine the "basic" components, first we solve for superspace torsions and curvatures in terms of supervielbeins and superconnections using the identifications (A.7) and (A.8); then we identify them with the off-shell solution to the Bianchi identities. The process is now standard [12]–[15]; we simply collect here the results.

$$\begin{aligned}
\overline{R}\big| &= 4\,(M + iN) && \text{(A.11)}\\
G_{A\dot{A}}\big| &= \frac{1}{3}A_{A\dot{A}} && \text{(A.12)}\\
\nabla_A\overline{R}\big| &= -4\psi_{mn}{}^{B}\sigma^{mn}{}_{AB} - 4i\,(M + iN)\,\psi_{A\dot{B}}{}^{\dot{B}} - 4iA^m\psi_{mA} && \text{(A.13)}\\
\nabla_{\underline{\dot{C}}}G_{A\underline{\dot{D}}}\Big| &= \frac{1}{4}\psi_{C\underline{\dot{C}}}{}^{C}{}_{\underline{\dot{D}}A} + \frac{i}{6}\,(M + iN)\,\psi_{A\underline{\dot{C}\dot{D}}} + \\
&\quad + \frac{i}{24}\Big(3A_{A\underline{\dot{C}}}\psi^{C}{}_{\underline{\dot{D}}C} - A_{C\underline{\dot{C}}}\psi_{A\underline{\dot{D}}}{}^{C} + 3A_{C\underline{\dot{C}}}\psi^{C}{}_{\underline{\dot{D}}A}\Big) && \text{(A.14)}\\
W_{ABC}\big| &= -\frac{1}{4}\psi_{\underline{A}}{}^{\dot{C}}{}_{\underline{B}\dot{C}\underline{C}} - \frac{i}{4}A_{\underline{A}}{}^{\dot{C}}\psi_{\underline{B}\dot{C}\underline{C}}\\
&= -\frac{1}{4}\psi_{mn\underline{A}}\sigma^{mn}_{\underline{BC}} - \frac{i}{4}A_m\psi_{n\underline{A}}\sigma^{mn}_{\underline{BC}} && \text{(A.15)}
\end{aligned}$$

With this knowledge, we can already compute $\nabla_m G_n|$, which is necessary to compute in x-space the terms we analyze in the main text:

$$\begin{aligned}
\nabla_m G_n| &= e_m^{\ \mu}\mathcal{D}_\mu\, G_n| - \frac{1}{2}\psi_m^A\, \nabla_A G_n| - \frac{1}{2}\psi_m^{\dot{A}}\, \nabla_{\dot{A}}G_n|\\
&= \frac{1}{3}e_m^{\ \mu}\mathcal{D}_\mu A_n + \frac{i}{12}\,(M - iN)\,\psi_m^A\psi_{nA} - \frac{i}{12}\,(M + iN)\,\psi_m^{\dot{A}}\psi_{n\dot{A}} + \\
&\quad + \frac{1}{6}\psi_{np}^{\dot{A}}\sigma^p_{\ A\dot{A}}\psi_m^A - \frac{i}{12}A_n\psi_p^{\dot{A}}\sigma^p_{\ A\dot{A}}\psi_m^A - \frac{1}{6}\psi_{np}^{A}\sigma^p_{\ A\dot{A}}\psi_m^{\dot{A}} - \frac{i}{12}A_n\psi_p^{A}\sigma^p_{\ A\dot{A}}\psi_m^{\dot{A}} - \\
&\quad - \frac{i}{24}\varepsilon_{npqs}\psi^{pq\dot{A}}\sigma^s_{\ A\dot{A}}\psi_m^A - \frac{1}{24}\varepsilon_{npqs}A^p\psi^{q\dot{A}}\sigma^s_{\ A\dot{A}}\psi_m^A - \\
&\quad - \frac{i}{24}\varepsilon_{npqs}\psi^{pqA}\sigma^s_{\ A\dot{A}}\psi_m^{\dot{A}} + \frac{1}{24}\varepsilon_{npqs}A^p\psi^{qA}\sigma^s_{\ A\dot{A}}\psi_m^{\dot{A}} && \text{(A.16)}
\end{aligned}$$

[2]Underlined indices are symmetrized with weigth one; undotted and dotted indices are always symmetrized independently.

To express $\nabla^2\overline{R}\big|$, $\nabla_{\underline{\dot{A}}}\nabla_{\underline{A}}G_{\underline{B\dot{B}}}\Big|$, $\nabla_{\underline{D}}W_{\underline{ABC}}\big|$, one must identify the (super)curvature $R_{\mu\nu}{}^{mn}$ with the x-space curvature $\mathcal{R}_{\mu\nu}{}^{mn}$, multiply by the inverse supervielbeins $E_M^{\ \mu}E_N^{\ \nu}$, identify with the solution to the Bianchi identities for R_{MN} and extract the field contents by convenient index contraction/symmetrization. The field content of these components will include the Riemann tensor in one of its irreducible components (respectively the Ricci scalar, the Ricci tensor and the Weyl tensor).

The result for $\nabla^2\overline{R}\big|$ is well known; it is necessary in order to compute the action for pure supergravity [13]:

$$
\begin{aligned}
\nabla^2\overline{R}\big| = {} & -8\mathcal{R} - \frac{32}{3}\left(M^2+N^2\right) - \frac{16}{3}A^m A_m + 16ie_m^\mu\mathcal{D}_\mu A^m + \\
& + 2\varepsilon^{mnrs}\sigma_s^{A\dot{A}}\psi_{mnA}\psi_{r\dot{A}} - 2\varepsilon^{mnrs}\sigma_s^{A\dot{A}}\psi_{mA}\psi_{nr\dot{A}} + \\
& + 8\left(M+iN\right)\psi_m^{\dot{A}}\psi_{\dot{A}}^m + 8A_n\sigma_m^{A\dot{A}}\psi_A^m\psi_{\dot{A}}^n - 16i\psi_{mn}^A\psi^{m\dot{A}}\sigma^n_{A\dot{A}}
\end{aligned}
\tag{A.17}
$$

The procedure for computing $\nabla_{\underline{\dot{A}}}\nabla_{\underline{A}}G_{\underline{B\dot{B}}}\Big|$, $\nabla_{\underline{D}}W_{\underline{ABC}}\big|$ is described in detail in appendix B of [16], but the full calculation, in terms of the supergravity multiplet, is just outlined. The results, expressed as functions of the known components, are

$$
\begin{aligned}
\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big| = {} & -i\,\nabla_{\underline{A\dot{A}}}G_{\underline{B\dot{B}}}\Big| + \frac{1}{8}\mathcal{R}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{AB}\sigma^{\rho\sigma}_{\dot{A}\dot{B}} + G_{\underline{A\dot{A}}}\Big|\,G_{\underline{B\dot{B}}}\Big| - \frac{1}{48}\sigma^{mn}_{AB}\psi_{m\underline{\dot{A}}}\psi_{n\underline{\dot{B}}}\,\overline{R}\big| + \\
& + \frac{1}{48}\sigma^{mn}_{\dot{A}\dot{B}}\psi_{m\underline{A}}\psi_{n\underline{B}}\,R\big| + \frac{1}{4}\sigma^{mn}_{AB}\psi_{m\underline{\dot{A}}}\psi_n^C\,G_{C\underline{\dot{B}}}\Big| + \frac{1}{4}\sigma^{mn}_{\dot{A}\dot{B}}\psi_{m\underline{A}}\psi_n^{\dot{C}}\,G_{\underline{B}\dot{C}}\Big| + \\
& + \frac{i}{48}\sigma^m_{\underline{A\dot{A}}}\psi_{m\underline{\dot{B}}}\,\nabla_{\underline{B}}\overline{R}\big| + \frac{i}{48}\sigma^m_{\underline{A\dot{A}}}\psi_{m\underline{B}}\,\nabla_{\underline{\dot{B}}}R\Big| - \frac{i}{8}\sigma^m_{\underline{A\dot{A}}}\psi_m^C\,\nabla_C G_{\underline{B\dot{B}}}\Big| - \\
& - \frac{i}{8}\sigma^m_{\underline{A\dot{A}}}\psi_m^C\,\nabla_{\underline{B}}G_{C\underline{\dot{B}}}\Big| - \frac{i}{8}\sigma^m_{C\underline{\dot{A}}}\psi_{m\underline{A}}\,\nabla^C G_{\underline{B\dot{B}}}\Big| - \frac{i}{8}\sigma^m_{C\underline{\dot{A}}}\psi_{m\underline{A}}\,\nabla_{\underline{B}}G^C{}_{\underline{\dot{B}}}\Big| - \\
& - \frac{i}{4}\sigma^m_{C\underline{\dot{A}}}\psi_m^C\,\nabla_{\underline{A}}G_{\underline{B\dot{B}}}\Big| - \frac{i}{8}\sigma^{m\ \dot{C}}_{\underline{A}}\psi_{m\underline{\dot{A}}}\,\nabla_{\underline{\dot{B}}}G_{\underline{B}\dot{C}}\Big| - \frac{i}{8}\sigma^{m\ \dot{C}}_{\underline{A}}\psi_{m\underline{\dot{A}}}\,\nabla_{\dot{C}}G_{\underline{B\dot{B}}}\Big| + \\
& + \frac{i}{4}\sigma^m_{\underline{A}\dot{C}}\psi_m^{\dot{C}}\,\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big| + \frac{i}{8}\sigma^m_{\underline{A\dot{A}}}\psi_m^{\dot{C}}\,\nabla_{\underline{\dot{B}}}G_{\underline{B}\dot{C}}\Big| + \frac{i}{8}\sigma^m_{\underline{A\dot{A}}}\psi_m^{\dot{C}}\,\nabla_{\dot{C}}G_{\underline{B\dot{B}}}\Big| + \\
& + \frac{i}{2}\sigma^{m\ \dot{C}}_{\underline{A}}\psi_{m\underline{B}}\,W_{\dot{A}\dot{B}\dot{C}}\big| - \frac{i}{2}\sigma^{mC}{}_{\underline{\dot{A}}}\psi_{m\underline{\dot{B}}}\,W_{ABC}\big|
\end{aligned}
\tag{A.18}
$$

$$
\begin{aligned}
\nabla_{\underline{A}}W_{\underline{BCD}}\big| = {} & -\frac{1}{8}\mathcal{R}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} - \frac{1}{24}\sigma^{mn}_{\underline{AB}}\psi_{m\underline{C}}\psi_{n\underline{D}}\,R\big| - \frac{1}{2}\sigma^{mn}_{\underline{AB}}\psi_m^{\dot{C}}\psi_{n\underline{C}}\,G_{\underline{D}\dot{C}}\Big| - \\
& - i\sigma^m_{\underline{A}\dot{C}}\psi_{m\underline{B}}\,\nabla_{\underline{C}}G_{\underline{D}}{}^{\dot{C}}\Big| + i\sigma^m_{\underline{A}\dot{C}}\psi_m^{\ \dot{C}}\,W_{\underline{BCD}}\big|
\end{aligned}
\tag{A.19}
$$

Due to the identity

$$
\varepsilon_{mnpq}\sigma^{pq}_{AB} = 2i\sigma_{mnAB}
\tag{A.20}
$$

the operator $\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}$ acts as an "antiself-dual projector":

$$
\mathcal{R}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} = \mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} =: -8\mathcal{W}_{ABCD}
\tag{A.21}
$$

$$
\mathcal{W}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{AB}\sigma^{\rho\sigma}_{\dot{A}\dot{B}} = 0
\tag{A.22}
$$

One can also show that

$$
\mathcal{R}_{\mu\nu}\sigma^\mu_{A\dot{A}}\sigma^\nu_{B\dot{B}} = \sigma^\mu_{\underline{A\dot{A}}}\sigma^\nu_{\underline{B\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \frac{1}{2}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{AB}\mathcal{R}
\tag{A.23}
$$

Defining $\mathcal{R}_{A\dot{A}B\dot{B}CD} = -\frac{1}{4}\mathcal{R}_{A\dot{A}B\dot{B}mn}\sigma^{mn}_{CD}$, the decomposition (A.1) can be written as [15]

$$\begin{aligned}\mathcal{R}_{A\dot{A}B\dot{B}CD} = &\frac{1}{8}\varepsilon_{\dot{A}\dot{B}}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \frac{1}{2}\varepsilon_{AB}\sigma^{\mu}_{\underline{C\dot{A}}}\sigma^{\nu}_{\underline{D\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) - \\ &- \frac{1}{12}\varepsilon_{\dot{A}\dot{B}}\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}\right)\mathcal{R} \end{aligned} \tag{A.24}$$

By extensively (and judiciously) using the equalities

$$\begin{aligned}\sigma^{m}_{C\dot{C}}\sigma^{n}_{\underline{A\dot{A}}}\sigma^{p}_{\underline{B\dot{B}}} &= \sigma^{n}_{C\dot{C}}\sigma^{m}_{\underline{A\dot{A}}}\sigma^{p}_{\underline{B\dot{B}}} + \varepsilon_{C\underline{A}}\sigma^{mn}_{\dot{A}\dot{B}}\sigma^{p}_{\underline{B}\dot{C}} - \varepsilon_{\dot{C}\underline{\dot{A}}}\sigma^{mn}_{AB}\sigma^{p}_{C\underline{\dot{B}}} + \\ &\quad + 2\varepsilon_{C\underline{A}}\varepsilon_{\dot{C}\underline{\dot{A}}}\left(\eta^{mp}\eta^{nq} - \eta^{mq}\eta^{np}\right)\sigma_{q\underline{B\dot{B}}} && \text{(A.25)}\\ \sigma^{mn}_{AB}\sigma^{p}_{C\dot{A}} + \sigma^{np}_{AB}\sigma^{m}_{C\dot{A}} + \sigma^{pm}_{AB}\sigma^{n}_{C\dot{A}} &= 2i\varepsilon^{mnpq}\varepsilon_{\underline{A}C}\sigma_{q\underline{B}\dot{A}} && \text{(A.26)}\\ \sigma^{mn}_{\dot{A}\dot{B}}\sigma^{p}_{A\dot{C}} + \sigma^{pm}_{\dot{A}\dot{B}}\sigma^{n}_{A\dot{C}} + \sigma^{np}_{\dot{A}\dot{B}}\sigma^{m}_{A\dot{C}} &= 2i\varepsilon^{mnpq}\varepsilon_{\underline{\dot{A}}\dot{C}}\sigma_{qA\underline{\dot{B}}} && \text{(A.27)}\end{aligned}$$

we carried the full computations of $\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big|$ and $\nabla_{\underline{A}}W_{\underline{BCD}}\Big|$, the results of which we now present (to our knowledge, for the first time in the literature):

$$\begin{aligned}\nabla_{\underline{A}}\nabla_{\underline{\dot{A}}}G_{\underline{B\dot{B}}}\Big| = &-\frac{1}{2}\sigma^{\mu}_{\underline{A\dot{A}}}\sigma^{\nu}_{\underline{B\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \frac{1}{9}A_m A_n\sigma^{m}_{\underline{A\dot{A}}}\sigma^{n}_{\underline{B\dot{B}}} - \frac{i}{3}e^{\mu}_{m}\mathcal{D}_{\mu}A_n\sigma^{m}_{\underline{A\dot{A}}}\sigma^{n}_{\underline{B\dot{B}}} - \\ &- \frac{1}{6}\left(M + iN\right)\sigma^{m}_{\underline{A\dot{A}}}\sigma^{n}_{\underline{B\dot{B}}}\psi^{\dot{C}}_{m}\psi_{n\dot{C}} - \frac{i}{3}\psi^{m}_{\underline{A}}\psi_{mn\underline{\dot{A}}}\sigma^{n}_{\underline{B\dot{B}}} - \frac{i}{3}\psi_{mn\underline{A}}\psi^{m}_{\underline{\dot{A}}}\sigma^{n}_{\underline{B\dot{B}}} - \\ &- \frac{i}{4}\sigma^{mn}_{\dot{A}\dot{B}}\sigma^{p}_{\underline{A}\dot{C}}\psi_{mn\underline{B}}\psi^{\dot{C}}_{p} + \frac{i}{12}\sigma^{np}_{\dot{A}\dot{B}}\sigma^{m}_{\underline{A}\dot{C}}\psi_{m\underline{B}}\psi^{\dot{C}}_{np} + \frac{i}{6}\sigma^{mn}_{AB}\sigma^{p}_{C\underline{\dot{A}}}\psi^{C}_{mn}\psi_{p\underline{\dot{B}}} + \\ &+ \frac{1}{12}\varepsilon_{mnpq}\psi^{mn}_{\underline{A}}\psi^{p}_{\underline{\dot{A}}}\sigma^{q}_{\underline{B\dot{B}}} - \frac{1}{12}\varepsilon_{mnpq}\psi^{m}_{\underline{A}}\psi^{np}_{\underline{\dot{A}}}\sigma^{q}_{\underline{B\dot{B}}} - \frac{i}{48}\varepsilon_{mnpq}A^{m}\psi^{n}_{\underline{A}}\psi^{p}_{\underline{\dot{A}}}\sigma^{q}_{\underline{B\dot{B}}} + \\ &+ \frac{1}{48}A_m\sigma^{m}_{\underline{A\dot{A}}}\psi_{n\underline{B}}\psi^{n}_{\underline{\dot{B}}} + \frac{3}{16}A_m\sigma^{n}_{\underline{A\dot{A}}}\psi_{n\underline{B}}\psi^{m}_{\underline{\dot{B}}} - \frac{7}{48}A_m\sigma^{n}_{\underline{A\dot{A}}}\psi^{m}_{\underline{B}}\psi_{n\underline{\dot{B}}} - \\ &- \frac{1}{6}\sigma^{mn}_{AB}\sigma^{p}_{C\underline{\dot{A}}}A_m\psi^{C}_{n}\psi_{p\underline{\dot{B}}} + \frac{1}{12}\sigma^{mp}_{\dot{A}\dot{B}}\sigma^{n}_{\underline{A}\dot{C}}A_m\psi_{n\underline{B}}\psi^{\dot{C}}_{p} + \frac{1}{4}\sigma^{mn}_{\dot{A}\dot{B}}\sigma^{p}_{\underline{A}\dot{C}}A_m\psi_{n\underline{B}}\psi^{\dot{C}}_{p} - \\ &- \frac{1}{6}\sigma^{C\dot{C}}_{m}\sigma^{n}_{\underline{A\dot{A}}}\sigma^{p}_{\underline{B\dot{B}}}A^{m}\psi_{nC}\psi_{p\dot{C}} && \text{(A.28)}\\ \nabla_{\underline{A}}W_{\underline{BCD}}\Big| = &-\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} - \frac{1}{4}\sigma^{mn}_{\underline{AB}}\sigma^{r}_{\underline{C}\dot{A}}A_m\psi_{n\underline{D}}\psi^{\dot{A}}_{r} - \frac{1}{4}\sigma^{mr}_{\underline{AB}}\sigma^{n}_{\underline{C}\dot{A}}A_m\psi_{n\underline{D}}\psi^{\dot{A}}_{r} - \\ &- \frac{i}{4}\sigma^{mr}_{\underline{AB}}\sigma^{n}_{\underline{C}\dot{A}}\psi_{n\underline{D}}\psi^{\dot{A}}_{mr} + \frac{i}{4}\sigma^{mn}_{\underline{AB}}\sigma^{r}_{\underline{C}\dot{A}}\psi_{mn\underline{D}}\psi^{\dot{A}}_{r} && \text{(A.29)}\end{aligned}$$

Knowing these components, we can compute, in x-space, any action which involves the supergravity multiplet. As an example, necessary for our purposes, we indicate how to compute some other derivatives of G_m which arise in the field content of some of the terms in section 3.

$$\begin{aligned}\nabla_{\dot{A}}G^2 &= G^{B\dot{B}}\nabla_{\dot{A}}G_{B\dot{B}} && \text{(A.30)}\\ \overline{\nabla}^2 G^2 &= \left(\nabla^{\dot{A}}G^{B\dot{B}}\right)\nabla_{\dot{A}}G_{B\dot{B}} - \frac{i}{6}G^{B\dot{B}}\nabla_{B\dot{B}}\overline{R} - \overline{R}G^2 && \text{(A.31)}\\ \nabla_A\overline{\nabla}^2 G^2 &= -2\left(\nabla^{\dot{A}}G^{B\dot{B}}\right)\nabla_A\nabla_{\dot{A}}G_{B\dot{B}} - \frac{i}{6}\left(\nabla_A G^{B\dot{B}}\right)\nabla_{B\dot{B}}\overline{R} - \frac{i}{6}G^{B\dot{B}}\nabla_A\nabla_{B\dot{B}}\overline{R} - \\ &\quad - G^2\nabla_A\overline{R} - \overline{R}G^{B\dot{B}}\nabla_A G_{B\dot{B}} && \text{(A.32)}\end{aligned}$$

$$
\begin{aligned}
\nabla^2\overline{\nabla}^2 G^2 = & -\frac{1}{1152}\left(\nabla^2\overline{R}\right)\nabla^2\overline{R} - 2\left(\nabla^A\nabla^{\dot A}G^{B\dot B}\right)\nabla_{\underline{A}}\nabla_{\underline{\dot A}}G_{\underline{B}\underline{\dot B}} - \frac{1}{3}G^2\nabla^2\overline{R} + \\
& + \frac{i}{6}\left(\nabla^{\underline{\dot C}}G^{A\underline{\dot D}}\right)\nabla_{A\dot D}\nabla_{\dot C}R - \frac{i}{3}G^m\nabla_m\nabla^2\overline{R} - \frac{i}{144}\left(\nabla_{\dot C}R\right)\nabla^{A\dot C}\nabla_A\overline{R} + \\
& + \frac{1}{3456}R\left(\nabla^A\overline{R}\right)\nabla_A\overline{R} + \frac{1}{96}G^{B\dot B}\left(\nabla_B\overline{R}\right)\nabla_{\dot B}R + \frac{1}{18}\left(\nabla^m\overline{R}\right)\nabla_m R - \\
& - \frac{7}{6}G^{B\dot B}\left(\nabla^A\overline{R}\right)\nabla_{\underline{A}}G_{\underline{B}\dot B} + \frac{7}{12}G_{B\dot B}\left(\nabla_{\dot A}\overline{R}\right)\nabla^{\underline{\dot A}}G^{B\underline{\dot B}} + \frac{2}{9}iRG^m\nabla_m\overline{R} - \\
& - \frac{i}{3}\overline{R}G^m\nabla_m R - \frac{i}{3}\left(\nabla^{\underline{A}}G^{\underline{B}\dot B}\right)\nabla_{B\dot B}\nabla_A\overline{R} - \frac{5}{3}R\left(\nabla^{\underline{\dot A}}G^{B\underline{\dot B}}\right)\nabla_{\dot A}G_{B\dot B} + \\
& + \overline{R}RG^2 + 8\left(\nabla^m G^n\right)\nabla_m G_n - 8\left(\nabla^m G^n\right)\nabla_n G_m + 8i\varepsilon^{mnrs}\left(\nabla_m G_n\right)\nabla_r G_s + \\
& + 8i\left(\nabla^{\underline{\dot A}}G^{B\underline{\dot B}}\right)\nabla^A{}_{\dot A}\nabla_{\underline{A}}G_{\underline{B}\dot B} - 12\left(\nabla^{\underline{\dot A}}G^{B\underline{\dot B}}\right)G^A{}_{\dot A}\nabla_{\underline{A}}G_{\underline{B}\dot B} + \\
& + 16W^{\dot A\dot B\dot C}G^B{}_{\dot C}\nabla_{\dot A}G_{B\dot B}
\end{aligned}
\tag{A.33}
$$

B. Detailed calculation of $\nabla_A W^2|$ and $\nabla^2 W^2|$ at tree level

Let's start by computing $\nabla_A W^2|$. From (2.6), we may write

$$
\nabla_A W^2\big| = -2\, W^{BCD}\big|\, \nabla_{\underline{A}} W_{\underline{BCD}}\big| + \mathcal{O}\left(\alpha\right)
\tag{B.1}
$$

which may be written, from (A.15) and (A.29), as

$$
\begin{aligned}
\nabla_A W^2\big| = & -\frac{1}{16}\mathcal{R}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}\psi^{rsB}\sigma^{CD}_{rs} - \frac{i}{8}\psi^{rsB}\psi_{mn\underline{D}}\psi_{p\dot A}\sigma^{CD}_{rs}\sigma^{mn}_{\underline{AB}}\sigma^{p}{}_{\underline{C}}{}^{\dot A} + \\
& + \frac{i}{8}\psi^{rsB}\psi_{p\underline{D}}\psi_{mn\dot A}\sigma^{CD}_{rs}\sigma^{mn}_{\underline{AB}}\sigma^{p}{}_{\underline{C}}{}^{\dot A} + \mathcal{O}\left(\alpha\right)
\end{aligned}
\tag{B.2}
$$

Let's consider first the terms in (2.11) which depend on the Riemann tensor. Using the expansion

$$
\begin{aligned}
\sigma^{mn}_{\underline{AB}}\sigma^{pq}_{\underline{CD}}\sigma^{rsAB} = & -\frac{2}{3}\sigma^{pq}_{CD}\left(\eta^{ms}\eta^{nr} - \eta^{mr}\eta^{ns}\right) - \frac{2}{3}\sigma^{mn}_{CD}\left(\eta^{ps}\eta^{qr} - \eta^{pr}\eta^{qs}\right) - \\
& - \frac{2}{3}\sigma^{rs}_{CD}\left(\eta^{mp}\eta^{nq} - \eta^{np}\eta^{mq}\right) - \frac{2}{3}i\varepsilon^{mnrs}\sigma^{pq}_{CD} - \frac{2}{3}i\varepsilon^{pqrs}\sigma^{mn}_{CD} - \\
& - \frac{i}{6}\varepsilon^{pqnr}\sigma^{ms}_{CD} + \frac{i}{6}\varepsilon^{pqmr}\sigma^{ns}_{CD} - \frac{i}{6}\varepsilon^{mnqr}\sigma^{ps}_{CD} + \frac{i}{6}\varepsilon^{mnpr}\sigma^{qs}_{CD} + \\
& + \frac{i}{6}\varepsilon^{pqns}\sigma^{mr}_{CD} - \frac{i}{6}\varepsilon^{pqms}\sigma^{nr}_{CD} + \frac{i}{6}\varepsilon^{mnqs}\sigma^{pr}_{CD} - \frac{i}{6}\varepsilon^{mnps}\sigma^{qr}_{CD} + \\
& + \frac{i}{6}\sigma_u{}^s{}_{CD}\left(\varepsilon^{pqmu}\eta^{nr} - \varepsilon^{pqnu}\eta^{mr} + \varepsilon^{mnpu}\eta^{qr} - \varepsilon^{mnqu}\eta^{pr}\right) - \\
& - \frac{i}{6}\sigma_u{}^r{}_{CD}\left(\varepsilon^{pqmu}\eta^{ns} - \varepsilon^{pqnu}\eta^{ms} + \varepsilon^{mnpu}\eta^{qs} - \varepsilon^{mnqu}\eta^{ps}\right)
\end{aligned}
\tag{B.3}
$$

those terms can be obtained from the corresponding one in (B.2). The terms in (2.11) which do not depend on the Riemann tensor can also be obtained from (B.2) using the

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expansion

$$
\begin{aligned}
\sigma^{mn}_{\underline{AB}}\sigma^{rsCD}\chi_{\underline{C}}\psi_{\underline{D}} = {} & \frac{1}{6}\sigma^{mn}_{AB}\sigma^{rsCD}\chi_C\psi_D - \left(\eta^{ms}\eta^{nr} - \eta^{mr}\eta^{ns} + i\varepsilon^{mnrs}\right)\chi_{\underline{A}}\psi_{\underline{B}} + \\
& + \frac{1}{6}\sigma^{ms}{}_{\underline{A}}{}^{D}\eta^{nr}\chi_{\underline{B}}\psi_D - \frac{1}{6}\sigma^{mr}{}_{\underline{A}}{}^{D}\eta^{ns}\chi_{\underline{B}}\psi_D + \frac{1}{6}\sigma^{ms}{}_{\underline{A}}{}^{D}\eta^{nr}\chi_D\psi_{\underline{B}} - \\
& - \frac{1}{6}\sigma^{mr}{}_{\underline{A}}{}^{D}\eta^{ns}\chi_D\psi_{\underline{B}} - \frac{1}{6}\sigma^{ns}{}_{\underline{A}}{}^{D}\eta^{mr}\chi_{\underline{B}}\psi_D + \frac{1}{6}\sigma^{nr}{}_{\underline{A}}{}^{D}\eta^{ms}\chi_{\underline{B}}\psi_D - \\
& - \frac{1}{6}\sigma^{ns}{}_{\underline{A}}{}^{D}\eta^{mr}\chi_D\psi_{\underline{B}} + \frac{1}{6}\sigma^{nr}{}_{\underline{A}}{}^{D}\eta^{ms}\chi_D\psi_{\underline{B}} + \frac{i}{6}\varepsilon^{mnr}{}_{t}\sigma^{ts}{}_{\underline{A}}{}^{D}\chi_{\underline{B}}\psi_D - \\
& - \frac{i}{6}\varepsilon^{mns}{}_{t}\sigma^{tr}{}_{\underline{A}}{}^{D}\chi_{\underline{B}}\psi_D + \frac{i}{6}\varepsilon^{mnr}{}_{t}\sigma^{ts}{}_{\underline{A}}{}^{D}\chi_D\psi_{\underline{B}} - \frac{i}{6}\varepsilon^{mns}{}_{t}\sigma^{tr}{}_{\underline{A}}{}^{D}\chi_D\psi_{\underline{B}}
\end{aligned}
\tag{B.4}
$$

This concludes our derivation of (2.11).

In order to compute $\nabla^2 W^2\big|$ we write, from (2.8),

$$
\nabla^2 W^2\big| = -2\,\nabla^{A}W^{BCD}\big|\,\nabla_{\underline{A}}W_{\underline{BCD}}\big| + \mathcal{O}\left(\alpha\right) \tag{B.5}
$$

which we may write, from (A.29), as

$$
\begin{aligned}
\nabla^2 W^2\big| = {} & -2\mathcal{W}^{ABCD}\mathcal{W}_{ABCD} + \frac{i}{8}\mathcal{R}_{\mu\nu\rho\sigma}\psi^{mD}\psi_{rs\dot{A}}\sigma^{C\dot{A}}_{m}\sigma^{rsAB}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} - \\
& - \frac{i}{8}\mathcal{R}_{\mu\nu\rho\sigma}\psi^{D}_{rs}\psi^{m}_{\dot{A}}\sigma^{C\dot{A}}_{m}\sigma^{rsAB}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \frac{1}{8}\sigma^{mn}_{\underline{AB}}\sigma^{pqAB}\sigma^{r}_{\underline{C}}{}^{\dot{A}}\sigma^{sC\dot{B}}\psi_{r\underline{D}}\psi_{mn\dot{A}}\psi^{D}_{s}\psi_{pq\dot{B}} + \\
& + \frac{1}{8}\sigma^{mn}_{\underline{AB}}\sigma^{pqAB}\sigma^{r}_{\underline{C}}{}^{\dot{A}}\sigma^{sC\dot{B}}\psi_{mn\underline{D}}\psi_{r\dot{A}}\psi^{D}_{pq}\psi_{s\dot{B}} - \\
& - \frac{1}{4}\sigma^{mn}_{\underline{AB}}\sigma^{pqAB}\sigma^{r}_{\underline{C}}{}^{\dot{A}}\sigma^{sC\dot{B}}\psi_{r\underline{D}}\psi_{mn\dot{A}}\psi^{D}_{pq}\psi_{s\dot{B}} + \mathcal{O}\left(\alpha\right)
\end{aligned}
\tag{B.6}
$$

From the decomposition (A.24) one can easily derive

$$
\mathcal{W}^{ABCD}\mathcal{W}_{ABCD} = \mathcal{W}^{\mu\nu\rho\sigma}_{+}\mathcal{W}^{+}_{\mu\nu\rho\sigma} \tag{B.7}
$$

Using the expansion (B.3), it is straightforward to check that

$$
\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma}\psi^{mD}\psi_{rs\dot{A}}\sigma^{C\dot{A}}_{m}\sigma^{rsAB}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} &= \frac{64}{3}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\psi^{\rho A}\psi^{\mu\nu\dot{A}}\sigma^{\sigma}_{A\dot{A}} \\
\mathcal{R}_{\mu\nu\rho\sigma}\psi^{D}_{rs}\psi^{m}_{\dot{A}}\sigma^{C\dot{A}}_{m}\sigma^{rsAB}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} &= \frac{64}{3}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\psi^{\mu\nu A}\psi^{\rho\dot{A}}\sigma^{\sigma}_{A\dot{A}}
\end{aligned}
\tag{B.8}
$$

Having these expressions, we get (2.12).

We include here the full σ-matrix expansion for the four-fermion terms; we did not include it in the main body to keep the text more readable:

$$
\begin{aligned}
& \sigma^{mn}_{AB}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{pq\underline{AB}}\sigma^{s\underline{C}\dot{B}}\varepsilon^{\underline{DE}} \\
& \quad = \frac{1}{12}\sigma^{mn}_{AB}\sigma^{pqAB}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sC\dot{B}}\varepsilon^{DE} + \frac{1}{12}\sigma^{mn}_{AB}\sigma^{pqAB}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sD\dot{B}}\varepsilon^{CE} + \\
& \qquad + \frac{1}{6}\sigma^{mn}_{AB}\sigma^{pqAC}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sB\dot{B}}\varepsilon^{DE} + \frac{1}{6}\sigma^{mn}_{AB}\sigma^{pqAD}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sC\dot{B}}\varepsilon^{BE} + \\
& \qquad + \frac{1}{6}\sigma^{mn}_{AB}\sigma^{pqCD}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sB\dot{B}}\varepsilon^{AE} + \frac{1}{6}\sigma^{mn}_{AB}\sigma^{pqAC}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sD\dot{B}}\varepsilon^{BE} + \\
& \qquad + \frac{1}{6}\sigma^{mn}_{AB}\sigma^{pqAD}\sigma^{r}_{C}{}^{\dot{A}}\sigma^{sB\dot{B}}\varepsilon^{CE}
\end{aligned}
$$

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$$
\begin{aligned}
= & -\frac{1}{3}\left(\eta^{mq}\eta^{np}-\eta^{mp}\eta^{nq}+i\varepsilon^{mnpq}\right)\left(\sigma^{rs\dot{A}\dot{B}}+\eta^{rs}\varepsilon^{\dot{A}\dot{B}}\right)\varepsilon^{DE}+ \\
& +\frac{1}{6}\left(\eta^{mq}\eta^{np}-\eta^{mp}\eta^{nq}+i\varepsilon^{mnpq}\right)\sigma^{rE\dot{A}}\sigma^{sD\dot{B}}- \\
& -\frac{1}{6}\left(\eta^{np}\sigma^{mqDE}-\eta^{mp}\sigma^{nqDE}+i\varepsilon^{mnpt}\sigma_{t}{}^{qDE}\right)\left(\eta^{rs}\varepsilon^{\dot{A}\dot{B}}+\sigma^{rs\dot{A}\dot{B}}\right)- \\
& -\frac{1}{6}\left(\eta^{mp}\eta^{ns}\eta^{rq}-\eta^{np}\eta^{ms}\eta^{rq}-\eta^{mq}\eta^{ns}\eta^{rp}+\eta^{nq}\eta^{ms}\eta^{rp}\right)\varepsilon^{\dot{A}\dot{B}}\varepsilon^{DE}- \\
& -\frac{i}{6}\left(\varepsilon^{mnsp}\eta^{rq}-\varepsilon^{mnsq}\eta^{rp}+\varepsilon^{pqrm}\eta^{ns}-\varepsilon^{pqrn}\eta^{ms}\right)\varepsilon^{\dot{A}\dot{B}}\varepsilon^{DE}- \\
& -\frac{1}{6}\left(\eta^{ms}\eta^{rq}\sigma^{np\dot{A}\dot{B}}+\eta^{ns}\eta^{rp}\sigma^{mq\dot{A}\dot{B}}-\eta^{ns}\eta^{rq}\sigma^{mp\dot{A}\dot{B}}-\eta^{ms}\eta^{rp}\sigma^{nq\dot{A}\dot{B}}\right)\varepsilon^{DE}- \\
& -\frac{i}{6}\left(\varepsilon^{mnsu}\eta^{rp}\sigma_{u}{}^{q\dot{A}\dot{B}}-\varepsilon^{mnsu}\eta^{rq}\sigma_{u}{}^{p\dot{A}\dot{B}}+\varepsilon^{pqru}\eta^{ms}\sigma_{u}{}^{n\dot{A}\dot{B}}-\varepsilon^{pqru}\eta^{ns}\sigma_{u}{}^{m\dot{A}\dot{B}}\right)\varepsilon^{DE}+ \\
& +\frac{1}{6}\left(\eta^{mu}\eta^{ns}-\eta^{ms}\eta^{nu}+i\varepsilon^{mnsu}\right)\left(\eta^{pv}\eta^{rq}-\eta^{rp}\eta^{qv}+i\varepsilon^{pqrv}\right)\sigma_{u}^{E\dot{A}}\sigma_{v}^{D\dot{B}}+ \\
& +\frac{1}{6}\left[\left(\eta^{mq}\eta^{rp}-\eta^{mp}\eta^{rq}+i\varepsilon^{mpqr}\right)\sigma^{nE\dot{A}}-\left(\eta^{nq}\eta^{rp}-\eta^{np}\eta^{rq}+i\varepsilon^{npqr}\right)\sigma^{mE\dot{A}}\right]\sigma^{sD\dot{B}}+ \\
& +\frac{1}{6}\left[\left(\eta^{ms}\eta^{np}-\eta^{mp}\eta^{ns}+i\varepsilon^{mnps}\right)\sigma^{qD\dot{B}}-\left(\eta^{nq}\eta^{ms}-\eta^{ns}\eta^{mq}+i\varepsilon^{mnqs}\right)\sigma^{pD\dot{B}}\right]\sigma^{rE\dot{A}}+ \\
& +\frac{i}{6}\left(\varepsilon^{mnpu}\eta^{qr}-\varepsilon^{mnqu}\eta^{pr}\right)\sigma_{u}^{E\dot{A}}\sigma^{sD\dot{B}}+\frac{i}{6}\left(\varepsilon^{mpqu}\eta^{ns}-\varepsilon^{npqu}\eta^{ms}\right)\sigma^{rE\dot{A}}\sigma_{u}^{D\dot{B}}+ \\
& +\frac{1}{6}\varepsilon^{mnuv}\varepsilon^{pqr}{}_{v}\sigma_{u}^{E\dot{A}}\sigma^{sD\dot{B}}+\frac{1}{6}\varepsilon^{mnsv}\varepsilon^{pqu}{}_{v}\sigma^{rE\dot{A}}\sigma_{u}^{D\dot{B}}+ \\
& +\frac{1}{6}\varepsilon^{mnsu}\varepsilon^{pqrv}\left(\eta_{uv}\varepsilon^{\dot{A}\dot{B}}-\sigma_{uv}^{\dot{A}\dot{B}}\right)\varepsilon^{DE}
\end{aligned}
\tag{B.9}
$$

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